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Engineering Analysis

3rd year

Gaussian elimination

An alternative technique for the solution of simultaneous equations is that of Gaussian elimination.

Ex: use Gaussian elimination to solve

$$2x + 3y = 1$$
$$x + y = 3$$

Solution

1- The augmented matrix is

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

2- use the row operations. The operations allowed to eliminate unwanted variables are:

a- any equation can be multiply by any non-zero constant.

b. any equation can be added to or subtracted from any other equation

c- equation can be interchanged.

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow[2R_2 - R_1]{R_1} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix}$$

The last line means $0x - 1y = 5 \Rightarrow y = -5$

$2x + 3 \times -5 = 1 \Rightarrow 2x = 16 \Rightarrow x = 8$

<u>Ex:</u> use Gaussian elimination to solve

$$2x + 3y = 4$$
$$4x + 6y = 7$$

<u>Solution</u>

The augmented matrix

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 7 \end{pmatrix} \xrightarrow[R_2 - 2R_1]{R_1} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 0 & -1 \end{pmatrix}$$

study of the last line seems to imply that $0x + 0y = -1$, which is clearly nonsense. The equations have no solution and we say that the simultaneous equations ar <u>inconsistent</u>.

Ex: use Gaussian elimination to solve ③

$$x + y = 0$$
$$2x + 2y = 0$$

Solution: the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right) \begin{array}{c} \xrightarrow{R_1} \\ \xrightarrow{R_2 - 2R_1} \end{array} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The last line $0x + 0y = 0$, This is not an inconsistency it is free variables.

Let $y = k$ where: k is any number.

$x + k = 0 \Rightarrow x = -k$.

There are thus an infinite number of solutions. for example

$x = -1 \quad y = 1$

or $x = \frac{1}{2} \quad y = -\frac{1}{2}$

and so on.

Observation of the coefficient matrices is the last two examples shows that they have a determinant of zero.

Ex: Solve by Gaussian elimination

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$$x - 4y - 2z = 21$$
$$2x + y + 2z = 3$$
$$3x + 2y - z = -2.$$

Solution The augmented matrix

$$\begin{pmatrix} 1 & -4 & -2 & 21 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & -1 & -2 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{\substack{R_1 \\ R_2 - 2R_1}} \begin{pmatrix} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 14 & 5 & -65 \end{pmatrix} \xrightarrow[R_3 - \frac{14}{9}R_2]{\substack{R_1 \\ R_2}}$$

$$\begin{pmatrix} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 0 & -\frac{13}{3} & \frac{-13}{3} \end{pmatrix}$$

$$\frac{-13}{3} z = \frac{-13}{3} \Longrightarrow z = 1$$

$$9y + 6z = -39 \Longrightarrow y = -5$$

$$x = 3$$

Ex: Solve the following equations by Gaussian elimin.

$$x - y + z = 3$$
$$x + 5y - 5z = 2$$
$$2x + y - z = 1$$

$$\begin{pmatrix} 1 & -1 & 1 & 3 \\ 1 & 5 & -5 & 2 \\ 2 & 1 & -1 & 1 \end{pmatrix} \xrightarrow[R_3-2R_1]{R_1,\ R_2-R_1} \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 6 & -6 & -1 \\ 0 & 3 & -3 & -5 \end{pmatrix} \xrightarrow[2R_3-R_2]{R_1,\ R_2}$$

$$\begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 6 & -6 & -1 \\ 0 & 0 & 0 & -9 \end{pmatrix}$$

The last line $0x + 0y + 0z = -9 \Longrightarrow$ it is inconsistent (no solution)

EX: solve the following equations by Gaussian elimination

$$2x - y + z = 2$$
$$-2x + y + z = 4$$
$$6x - 3y - 2z = -9$$

<u>solution</u>

$$\begin{pmatrix} 2 & -1 & 1 & 2 \\ -2 & 1 & 1 & 4 \\ 6 & -3 & -2 & -9 \end{pmatrix} \xrightarrow[R_3-3R_1]{R_1,\ R_2+R_1} \begin{pmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -5 & -15 \end{pmatrix} \xrightarrow[2R_3+5R_2]{R_1,\ R_2}$$

$$\begin{pmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad 2z = 6 \Longrightarrow z = 3$$

let $y = k$

$$2x - k + 3 = 2 \Longrightarrow x = \frac{1}{2}(-1 + k) = \frac{1}{2}(k-1)$$

Finiding the inverse matrix using row operations

A similar technique can be used to find the inverse of a square matrix A where this exists. suppose we are given the matrix A and wish to find its inverse B. Then we know $\boxed{A \cdot B = I}$

or $\boxed{A \cdot A^{-1} = I}$

that is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we form the augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{pmatrix}$$

Ex: Find the inverse matrix by row reduction to the identity

$$A = \begin{pmatrix} -1 & 8 & -2 \\ -6 & 49 & -10 \\ -4 & 34 & -5 \end{pmatrix}$$

Solution

$$\left(\begin{array}{ccc|ccc} -1 & 8 & -2 & 1 & 0 & 0 \\ -6 & 49 & -10 & 0 & 1 & 0 \\ -4 & 34 & -5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\substack{R_2-6R_1 \\ R_3-4R_1}]{R_1} \left(\begin{array}{ccc|ccc} -1 & 8 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -6 & 1 & 0 \\ 0 & 2 & 3 & -4 & 0 & 1 \end{array}\right) \xrightarrow[\substack{R_2 \\ R_3-2R_2}]{R_1-8R_2}$$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & -18 & 49 & -8 & 0 \\ 0 & 1 & 2 & -6 & 1 & 0 \\ 0 & 0 & -1 & 8 & -2 & 1 \end{array}\right) \xrightarrow[\substack{R_2+2R_3 \\ R_3}]{R_1-18R_3} \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -95 & 28 & -18 \\ 0 & 1 & 0 & 10 & -3 & 2 \\ 0 & 0 & -1 & 8 & -2 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \\ -R_3}]{-R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 95 & -28 & 18 \\ 0 & 1 & 0 & 10 & -3 & 2 \\ 0 & 0 & 1 & -8 & 2 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 95 & -28 & 18 \\ 10 & -3 & 2 \\ -8 & 2 & -1 \end{pmatrix}$$

Solution to systems of linear homogeneous equations

Consider the simultanous linear homogenous equations

$$ax + by = 0$$
$$cx + dy = 0$$

where a, b, c and d are constants. Clearly $x = 0$, $y = 0$ is a solution. It is called the trivial solution. Non-trivial solutions are solutions other than $x = 0$, $y = 0$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A \cdot V = 0$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $V = \begin{bmatrix} x \\ y \end{bmatrix}$, $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

If $|A| = 0$, the system has non-trivial solutions

If $|A| \neq 0$, the system has trival solutions

Ex: Decide which of the following systems of equations has non-trivial solutions.

a- $3x + 7y = 0$
$2x - y = 0$

b- $2x + y = 0$
$6x + 3y = 0$

Solution

a- we write the system as

$$\begin{pmatrix} 3 & 7 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

let $A = \begin{pmatrix} 3 & 7 \\ 2 & -1 \end{pmatrix}$

$|A| = -3 - 14 = -17$, the system has only the trivial solution.

b- we write the system as

$$\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

let $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$

$|A| = 6 - 6 = 0$, the system has non-trivial solutions

Example: Determine which of the following systems of equations has non-trivial solutions.

(a) $2x + y - 3z = 0$
$x - 3y + 2z = 0$
$5x - 8y + 3z = 0$

(b) $2x + y - 3z = 0$
$x - 3y + 2z = 0$
$5x - 7y + 3z = 0$

Solution:

a- we have $A \cdot V = 0$

where

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & -3 & 2 \\ 5 & -8 & 3 \end{pmatrix} \qquad V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Evaluation of $|A|$ shows that $|A| = 0$ and so the system has non-trival solutions

b - Here

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & -3 & 2 \\ 5 & -7 & 3 \end{pmatrix}$$

from which $|A| = -7$. since $|A| \neq 0$. the system has trivial solution.

Eigenvalues

we will explain the meaning of the term eigenvalues by means of an example - consider the system

$$2x + y = \lambda x$$
$$3x + 4y = \lambda y$$

Where λ is some unknown constant. clearly these equs have the trivial solution $x=0, y=0$. The equations may be Written in the matrix form as

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

or, using the usual notation,

$$AV = \lambda V$$

we now seek values of λ so that the system has non-trivial solution

$$AV - \lambda V = 0$$
$$(A - \lambda)V = 0$$

the form $(A - \lambda)$ is not defined. A is a matrix and λ is constant. To help us do this, we use the (2×2) identity matrix, I.

$$\lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot I \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \lambda I V$$

$\therefore \quad A \cdot V = \lambda I V$

$AV - \lambda I V = 0 \Rightarrow (A - \lambda I) V = 0$

Note that the expression $(A - \lambda I)$ is defined since both A and (λI) are square matrices of the same size.

for $(A - \lambda I) V = 0$ to have non-trivial solution requires $|A - \lambda I| = 0$ It is called the characteristic equation

Now, $|A - \lambda I| = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{pmatrix}$

so the condition $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(4-\lambda) - 3 \times 1 = 0$

$\lambda^2 - 6\lambda + 5 = 0$

$(\lambda - 1)(\lambda - 5) = 0 \Rightarrow \lambda = 1, \lambda = 5$

These are the values of λ which cause the system $AV = \lambda V$ to have non-trivial solution

They are called eigenvalues

EX: Find the values of λ for which

$$x + 4y = \lambda x$$
$$2x + 3y = \lambda y$$

has non-trivial solutions.

Solution We write the system as

$$A \cdot V = \lambda \cdot V$$

where $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, $V = \begin{pmatrix} x \\ y \end{pmatrix}$

To have non-trivial solutions, we require

$$|(A - \lambda I)| = 0$$

$$(A - \lambda I) = \left(\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$
$$= \left(\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = \begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix}$$

$$\left| \begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} \right| = 0 \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$3 - 4\lambda + \lambda^2 - 8 = 0$$
$$\lambda^2 - 4\lambda - 5 = 0$$
$$(\lambda - 5)(\lambda + 1) = 0$$
$$\lambda = 5 \text{ or } \lambda = -1$$

The given system has non-trivial solution when $\lambda = -1$ and $\lambda = 5$.
These are the eigenvalues.

Ex: Determine the characteristic equation and eigenvalues λ, in the system

$$\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

Solution: The characteristic equation is given by

$$\left|\left(A - \lambda I\right)\right| = 0$$

$$\left|\left(\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right| = 0 \Rightarrow \left|\left(\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)\right|$$

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 5-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)(\lambda - 4) = 0$$

$$\lambda = 4 \text{ (twice)}$$

There is one repeated eigenvalue, $\lambda = 4$

Ex: Find ⓐ the characteristic equation ⓑ the eigenvalues of A where

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix}$$

Solution

a- $|(A-\lambda I)| = 0$

$$(A-\lambda I) = \begin{pmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 3 & 2 & -2-\lambda \end{pmatrix}$$

and

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 3 & 2 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(-1-\lambda)(-2-\lambda)-2\right]-2\left[(2+\lambda)-3\right]$$

$$-\lambda^3-2\lambda^2+\lambda+2=0 \implies \lambda^3+2\lambda^2-\lambda-2=0$$

The characteristic equation $\boxed{\lambda^3+2\lambda^2-\lambda-2=0}$.

b- $\lambda^3+2\lambda^2-\lambda-2=0$

$(\lambda+2)(\lambda+1)(\lambda-1)=0$

from which $\lambda=-2,-1,1$.

The eigenvalues are $\lambda=-2,-1,1$.

Exercises

1- By expressing the following equations in matrix form and use Gaussian elimination, solve

a) $4x - 2y = 14$
$2x + y = 5$

b) $2x - 2y = 0$
$x + 3y = -8$

c) $4x + y + 3z = 20$
$2x - y + 4z = 20$
$y + 5z = 20$

d) $4x + 3y - z = 6$
$x - 3z - y = -3$
$x + y + 2z = 4$

ans. a) $x = 3, y = -1$ b) $x = -2, y = -2$
c) $x = 2, y = 0, z = 4$ d) $x = y = z = 1$

2. Solve the following equations by Gaussian elimination:

a) $2x + y - 3z = -5$
$x - y + 2z = 12$
$7x - 2y + 3z = 37$

b) $x + y - z = 1$
$3x - y + 5z = 3$
$7x + 2y + 3z = 7$

c) $4x + 7y + 8z = 2$
$5x + 8y + 13z = 0$
$3x + 5y + 7z = 1$

d) $x + y + z = 7$
$x - y + 2z = 9$
$2x + y - z = 1$

ans. a) $x = 3, y = -5, z = 2$ b) $x = 1 - k, y = 2k, z = k$
c) Inconsistent d) $x = 2, y = 1, z = 4$

4-. Determine which of the following systems have non-trivial solutions:

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(a) $x+2y-z=0$
$3x+y+2z=0$
$x+y=0$

(b) $2x-3y-2z=0$
$3x+y-3z=0$
$x-7y-z=0$

(c) $x+2y+3z=0$
$4x-3y-z=0$
$6x+y+3z=0$

(d) $x+3z=0$
$x-y=0$
$y+2z=0$

(e) $x-2y=0$
$3x-6y=0$

(f) $3x+y=0$
$9x+2y=0$

(g) $4x-3y=0$
$-4x+3y=0$

(h) $6x-2y=0$
$2x-\frac{2}{3}y=0$

ans.

(a), (b), (e), (g) and (h) have non-trivial solution

5.- Calculate (i) the characteristic equation (ii) the eigenval of the system $A \cdot V = \lambda \cdot V$ where A is given by

(a) $\begin{pmatrix} 2 & 6 \\ 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & -2 \\ 1 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 3 \\ 4 & -1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & -1 & 2 \\ -3 & -2 & 3 \\ 2 & -1 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{pmatrix}$

(g) $\begin{pmatrix} 2 & 1 & 2 \\ -1 & 1 & -1 \\ 8 & 3 & 0 \end{pmatrix}$ (h) $\begin{pmatrix} -2 & 6 & 2 \\ 0 & 3 & 4 \\ 3 & -3 & 5 \end{pmatrix}$ (k) $\begin{pmatrix} 3 & -2 & 1 \\ 2 & -4 & 3 \\ 16 & -4 & 1 \end{pmatrix}$

ans.

(a) (i) $\lambda^2 - 6\lambda - 7 = 0$
(ii) $\lambda = -1, 7$

(b) (i) $\lambda^2 - 2\lambda + 1 = 0$
(ii) $\lambda = 1$ (twice)

(c) (i) $\lambda^2 - 11\lambda + 30 = 0$
(ii) $\lambda = 5, 6$

(d) (i) $\lambda^2 - 13 = 0$
(ii) $\lambda = -\sqrt{13}, \sqrt{13}$

(e) (i) $-\lambda^3 + 7\lambda + 6 = 0$
(ii) $\lambda = -2, -1, 3$

(f) (i) $\lambda^3 - 4\lambda^2 - 5\lambda + 12 = 0$
(ii) $\lambda = -\sqrt{3}, \sqrt{3}, 4$

(g) (i) $\lambda^3 - 3\lambda^2 - 10\lambda + 24 = 0$
(ii) $\lambda = -3, 2, 4$

(h) $\lambda^3 - 6\lambda^2 + 5\lambda = 0$
(ii) $\lambda = 0, 1, 5$

(k) (i) $\lambda^3 - 13\lambda + 12 = 0$
(ii) $\lambda = -4, 1, 3$

CAYLEY-HAMILTON THEORM

Every square matrix satisfies its own characteristic equation.
If $|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n)$ be the characteristic polynomial of $n\times n$ matrix $A=(a_{ij})$, then the matrix equation

$X^n + a_1X^{n-1} + a_2X^{n-2} + \dots + a_nI = 0$ is satisfied by $X=A$ i.e,

$A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI = 0$

Ex: Find the characteristic equation and A^{-1} of the matrix A

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution: The characteristic equation is $|(A-\lambda I)| = 0$

$$\left|\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}\right| = 0$$

$(2-\lambda)(1-\lambda) - 1 = 0 \Rightarrow 2 - 2\lambda - \lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 3\lambda + 1 = 0$ c/c equation

$A^2 - 3A + I = 0$ (Multiplying by A^{-1})

$A - 3I + A^{-1} = 0$

$A^{-1} = 3I - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

Ex: Find the characteristic equation of the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and verify that it is satisfied by A and hence obtain A^{-1}. Express $A^6-6A^5+9A^4-2A^3-12A^2+23A-9I$ in linear polynomial in A.

Solution characteristic equation $|A-\lambda I|=0$

$$\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}=0$$

$$\left[(2-\lambda)(2-\lambda)^2-1\right]+1\left[-2+\lambda+1\right]+1\left[1-2+\lambda\right]=0$$

$$(2-\lambda)^3-(2-\lambda)+\lambda-1+\lambda-1=0 \Rightarrow (2-\lambda)^3+3\lambda-4=0$$

$$8-\lambda^3-12\lambda+6\lambda^2+3\lambda-4=0$$

$$-\lambda^3+6\lambda^2-9\lambda+4=0 \Rightarrow \lambda^3-6\lambda^2+9\lambda-4=0$$

By cayley-Hamilton Theorem $A^3-6A^2+9A-4I=0$ ----(i)

Verification:

$$A^2=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}=\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 9A - 4I = 0$ (21)

$$\begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

So it is verified that the characteristic equation(1) is satisfied by A.

Inverse of Matrix A

$$A^3 - 6A^2 + 9A - 4I = 0 \qquad \text{Multiplying by } A^{-1}$$

$$A^2 - 6A - 9I - 4A^{-1} = 0$$

$$4A^{-1} = A^2 - 6A - 9I$$

$$4A^{-1} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

$$A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I = A^6 - 6A^5 + 9A^4 - 4A^3 + 2A^3 - 12A^2 + 18A + 5A + I$$

$$= A^3(A^3 - 6A^2 + 9A - 4I) + 2(A^3 - 6A + 9A + 4I) + 5A + I$$

$$= A^3 \cdot 0 + 2 \cdot 0 + 5A + I = 5A + I$$

Ex: The matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ is given. The eigenvalues of 22

$4A^{-1}+3A+2I$ are (A) 6,15 (B) 9,12 (C) 9,15 and (D) 7,15

<u>Solution</u> $|A-\lambda I| = 0$

$$\left|\begin{pmatrix} 1-\lambda & 0 \\ 2 & 4-\lambda \end{pmatrix}\right| = 0 \Rightarrow (1-\lambda)(4-\lambda) = 0$$

$\lambda^2 - 5\lambda + 4 = 0$ c/c equation.

$(A^2 - 5A + 4I = 0)$ multiplying by A^{-1}

$$A - 5I + 4A^{-1} = 0 \Rightarrow 4A^{-1} = 5I - A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}.$$

$$4A^{-1} = \begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix}$$

Let $B = 4A^{-1}+3A+2I = 4\times\left[\frac{1}{4}\begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix}\right] + 3\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$B = \begin{pmatrix} 9 & 0 \\ 4 & 15 \end{pmatrix}$$

to find the eigenvalues of B

$$|(B-\lambda I)| = 0 \Rightarrow \left|\begin{pmatrix} 9-\lambda & 0 \\ 4 & 15-\lambda \end{pmatrix}\right| = 0$$

$$(9-\lambda)(15-\lambda) = 0$$

$$\therefore \lambda = 9, \quad \lambda = 15$$

Exercise

1- Find the characteristic polynomial (equation) of three follow materices. verify Cayley Hamilton Theorem for this matrices. Hence find A^{-1}.

a- $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ b- $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

ans. (a) $A^{-1} = \frac{1}{20}\begin{bmatrix} 7 & -2 & -3 \\ 1 & 4 & 1 \\ -2 & 2 & 8 \end{bmatrix}$ (b) $A^{-1} = \frac{1}{11}\begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$

2- Use Cayley Hamilton Theorem to obtain the inverse of the following matrices

a- $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ b- $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{bmatrix}$ c- $A = \begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$

ans. (a) $A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (b) $A^{-1} = -\frac{1}{5}\begin{bmatrix} 4 & -5 & -2 \\ 7 & -10 & -1 \\ -2 & 0 & 1 \end{bmatrix}$ (c) $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 5 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

3- Show that the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation. Hence find A^{-1}

ans. $A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & 2 & -10 \\ -2 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$

4- If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, then express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.

ans $A + 5I$

5- If λ_1, λ_2 and λ_3 are the eigenvalues of the matrix

$$\begin{bmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{bmatrix}$$ then $\lambda_1+\lambda_2+\lambda_3$ is equal to

(i) -16 (ii) 2 (iii) -6 (iv) -14 ans. (ii)

6- The matrix $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, find the matrix A^{32} using Cayley Hamilton Theorem.

7- The matrix A is defined as $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$, find the eigenvalues of $3A^3+5A^2+6A+I$

ans. $15, -15, -53$

8- Find the eigenvalues of the matrix $A=\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

ans. $0, 1, -2$.

END OF LECTURE

ANY QUESTIONS?